Qualitative Dynamical Analysis of Queueing Networks with Inhibition

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ABSTRACT

An approximate dynamical extension of queueing theory result is described and is applied to Jackson's and G's Networks. It is shown that the dynamics of these networks may be represented by non-linear compartmental systems which are cooperative for the former case and may sometimes be competitive for the latter case. The implications in term of stability are discussed and an illustrative example is provided.

1. INTRODUCTION

Queueing theory first celebrated success about a century ago when Erlang modelled early telecommunication networks. This has since proved successful for the analysis and performance modelling of distributed computing systems. Recently, Gelenbe has proposed queueing models for auctions and viruses propagation [8]. These two subjects are receiving a lot of attention from the computer science community, the former, for its ability to produce outcome with good properties in system of self-interested agents [3] and the latter for the obvious nuisance brought about by computer viruses. A drawback of queueing theory is however its relative inability to deal with transient phenomena. Chapman-Kolmogorov equations describe the non-steady state of queueing systems but are usually very complex. On the other hand, dynamical systems are often used to represent complex dynamical behaviours. In particular, compartmental systems, governed by mass balance and flux balance equations, are used in many scientific fields (e.g. [13]), including population dynamics and bioprocess modelling. The Chapman-Kolmogorov system itself is, in fact, compartmental [4] and describes the balance of probability flow around each possible state. The main contribution of this paper is to show how the dynamical behaviour of some large classes of queueing systems may be represented approximately in a simplified manner using non-linear compartmental systems. The fact that this simplification is possible has been described before in the literature [20] however the stability of the approximated system was not discussed and the compartmental nature of the resulting system was not mentioned. Furthermore, our second contribution is to extend this result to G-Networks [6] where output of some queues may destroy the work accumulated in other queues. This is important as this mechanism introduces inhibition and competition in queueing networks[5]. Compartmental systems are often studied in conjunction with the theory of competitive and cooperative systems [19, 14]. The relationship that exists between this terminology and some queueing systems will be illustrated. In Section 2, the principle of our approximation is presented. In Section 3, the approximation is applied to Jackson's networks and the resulting system is shown to be cooperative and globally stable. In Section 4, the approximation is applied to G-Networks where it is shown that the approximate version is not necessarily cooperative and may be competitive in some cases. A simple example of a competitive system is given to illustrate our ideas. The transient response of this simple system is shown to exhibit a large overshoot which is reproduced by the approximation which is usually of practical interest. Finally a correction term is proposed to improve the approximation.

Approximate models such as those presented in this paper have been used in [20] and [10] to derive optimal control laws and in [17] to derive an adaptive feedback controller and in [11] to prove the stability of a non-linear hop-by-hop feedback controller. The link with queueing theory may therefore offer a possible microscopic model for some of these systems. In turn, dynamical systems may be used to analyse and control unwanted oscillations or large overshoots in networks which is hardly achieved with queueing theory alone.

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2. M/M/1 QUEUE

Let us first consider an M/M/1 queue with arrival rate λ and service rate μ. By definition of the exponential distribution and for any small Δt, the probability of an arrival and of a departure during any time interval $[t, t + \Delta t]$ may respectively be written

$$\begin{aligned} P(arrival) &= \lambda \Delta t + \mathcal{O}(\Delta t) \\ P(departure) &= \mu \Delta t + \mathcal{O}(\Delta t) \end{aligned}$$

where $\mathcal{O}(t)$ is such that $\lim_{t\to 0} \mathcal{O}(t)/t = 0$. The state of the system at any point in time t is the number of customers in the system, either waiting in the queue or in service and is represented by a r.v. denoted $K(t)$. Defining $p(k,t) = P(K(t) = k)$ with $k \in \mathbb{N}$ a possible value of the state, we therefore obtain

$$\begin{aligned} p(k, t+\Delta t) &= p(k,t)(1-\lambda\Delta t - \mu\Delta t) \\ &\quad + p(k-1,t)\lambda\Delta t \mathbf{1}(k>0) \\ &\quad + p(k+1,t)\mu\Delta t \end{aligned}$$

where terms in $\mathcal{O}(\Delta t)$ have been neglected and where $\mathbf{1}(x)$ is equal to one when x is true and zero otherwise. This leads to

$$\begin{aligned} \frac{\partial p(k,t)}{\partial t} =& -(\lambda+\mu)p(k,t) + \lambda p(k-1,t)\mathbf{1}(k>0) \\ & + \mu p(k+1,t) \end{aligned} \quad (1)$$

which is a linear, compartmental system of ordinary differential equations [4]. Notice that the state of this system, as defined in system theory is given by the vector $[p(0,t),\ldots,p(k,t),\ldots]^T$ and is different from the state $K(t)$ of the M/M/1 system as defined in queueing theory. Furthermore, note that the dimension of the system is infinite (but countable).

Queueing theory is typically concerned with the stationary solution of (1) given by

$$p(k) = \lim_{t\to+\infty} p(k,t) = q^k(1-q) \quad q = \frac{\lambda}{\mu} \quad (2)$$

and exists *iff* $q < 1$.

However, in many practical applications of system engineering one is interested not only in stationary solutions but also in the transient dynamics. In addition, from a practical point of view, it would be tempting to analyse an M/M/1 queue as a one dimensional system rather than the infinite dimensional system (1). In other words, one would like to obtain a one dimensional differential equation which describes the evolution of the average occupancy, or possibly an approximate version of the quantity

$$\bar{k}(t) = \sum_{k=0}^{+\infty} kp(k,t) \quad (3)$$

An elegant solution to this problem is found in [20] and is extensively applied in [9] : multiplying both sides of (1) by k and taking the sum over all possible values, one obtains

$$\frac{\partial \bar{k}(t)}{\partial t} = \lambda - (1-p(0,t))\mu \quad (4)$$

where $1 - p(0,t)$ is the probability that the server is busy at time t and which cannot be written as a function of $\bar{k}(t)$ only. The approximation therefore consists of expressing the probability that the server is busy as a function of the average buffer occupancy such that, at equilibrium, the exact value of $\bar{k}$ is recovered. Using $x(t)$ to denote the approximate average evolution, the approximate model is written

$$\dot{x} = \lambda - r(x)\mu \quad \textit{with} \quad r(x) = \frac{x}{1+x} \quad (5)$$

so that at equilibrium

$$r(x^*) = \frac{x^*}{1+x^*} = q \quad \textit{and} \quad x^* = \lim_{t\to+\infty} \bar{k}(t) = \frac{q}{1-q}$$

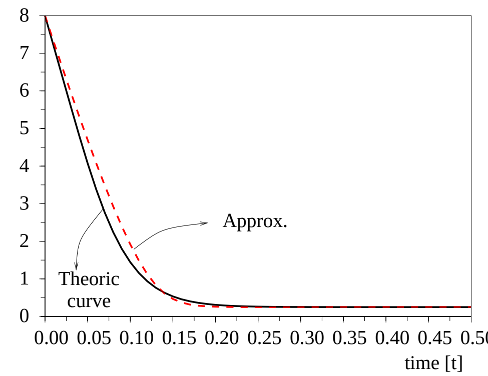

Figure 1: Comparison between the solution of (5) with $x(t=0) = 8$ (dashed line)and the solution of (1) with $p(8,0) = 1$ (the average computed with (3) is shown).

Fig. 1 shows a comparison between the solution of (5) with $x(t=0) = 8$ and the computation of (3) resulting from the solution[1] of (1) with $p(8,0) = 1$. The parameters of the queue are $\mu = 100$ and $\lambda = 20$. It can be seen that the trajectories are indeed very similar. However, as it will be argued later, an approximate dynamical analysis focuses on the reproduction of a qualitative behaviour rather than on the reproduction of exact numerical results. Indeed, results obtained in dynamical system theory are often closed loop and adaptive which makes them robust to modelling inadequacy as long as the overall dynamics are well represented.

3. OPEN JACKSON'S NETWORK OF M/M/1 QUEUES

A Jackson's network consists of the interconnection of a certain number N of M/M/1 queues [2] so that, when a customer finishes treatment in queue i, it heads towards queue j with probability p^+_{ij} ($p^+_{ii} = 0$) or leaves the network with probability d_i ($\sum_{j=1}^N p^+_{ij} + d_i = 1$). Customer arrivals from the outside world toward queue i form a Poisson process with parameter Λ_i. The service time of each queue is exponentially distributed with average μ_i. The system state is denoted $K(t) = (K_1(t),\ldots,K_N(t))^T$ where $K_i(t)$ represents the number of customers in queue i at time t.

A major result, known as Jackson's theorem is that such a network satisfies the product form property. It means that the equilibrium probability distribution of the state is given by

$$p(k) = \prod_{i=1}^N q_i^{k_i}(1-q_i)$$

[1]in practice, the M/M/1 system is approximated by a M/M/1/K system and K is chosen large enough so that convergence toward the desired solution is achieved

[2]in general, Jackson's networks may be composed of M/M/c queues but the discussion is limited here to the case where $c = 1$

where the q_i's are solutions of the linear system

$$0 = -\mu_i q_i + \sum_{j=1, i\neq j}^{N} \mu_j q_j p_{ji}^{+} + \Lambda_i \quad i = 1, \ldots, N \qquad (6)$$

provided that $q_i < 1 \forall i$. Equation (6) is known as the traffic equation. This result means that in a network of M/M/1 queues (even in a non-feedforward network), each queue may be considered as an isolated M/M/1 queue provided that the parameters of each queue have been resolved using the traffic equation (6). It means that the equilibrium joint distribution of the numbers of customers in each queue is the same as if the queues were independent M/M/1 queues with appropriate arrival rates. It is therefore not surprising that the dynamical approximation described in the previous section may be generalised to Jackson's networks as follows : define the matrix $G^J(x) = [g_{ij}]$[3] with

$$g_{ij}(x) = \mu_j p_{ji}^{+} \tilde{r}(x_j) \quad \textit{and} \quad g_{ii} = -\mu_i \tilde{r}(x_i)$$

with $\tilde{r}(x) = 1/(1+x)$, then the non-linear compartmental system

$$\dot{x} = f(x) = G^J(x)x + \lambda \qquad (7)$$

is an approximate dynamical extension of the stationary queueing theory results in the sense that the equilibrium solution of (7) exists *iff* the network is stable and is identical to the theoretical results given by queueing theory. Indeed, the equilibrium point x^* of (7) is obtained by solving

$$0 = G^J(x^*)x^* + \lambda$$

which becomes identical to (6) when posing $r(x_i^*) = q_i$. As $r(x)$ is strictly increasing and strictly bounded by one, there exists a unique equilibrium point *iff* $q_i < 1 \forall i$.

Equation (7) is a mass balance equation and states that the variation of average accumulated quantities in each queue is the difference between the total average input rate and the total output rate. These systems are known in the literature as *compartmental systems* and the matrix G^J is said to be *compartmental*. More formally, a compartmental matrix is a matrix with the following three properties :

- Diagonal entries are non-positive
- Non-diagonal entries are non-negative (also known as Metzler property)
- The matrix is column-wise diagonally dominant

Indeed, it is easy to verify that $g_{ii} = -\mu_i \tilde{r}(x_i) \leqslant 0$, $g_{ij} = \mu_j p_{ji}^{+} \tilde{r}(x_j) \geqslant 0$ and $\sum_{i=1}^{N} g_{ij} = -d_j \mu_j \tilde{r}(x_j) \leqslant 0$.

Compartmental systems are used in biology, population dynamics, chemistry, ecology and economics. They have strong structural properties and they have been analysed extensively [13, 1, 15, 16]. However, stability analysis of these non-linear systems remains a challenging issue. Stability results associated with compartmental systems are often drawn from the theory of positive and monotone systems.

Indeed, systems with a Metzler Jacobian matrix (Non-diagonal entries are non-negative) are well studied and are known as *cooperative* systems (also known as *monotone*). An important result related to cooperativity is as follows[19, 12] (see also [14])

[3]The notation G^J expresses the association of the compartmental matrix with the Jackson's network.

Property 1 (Hirsch). *Every bounded trajectories of a strongly monotone system converges to the set of equilibria.*

This property holds almost everywhere (i.e. there might exists a set of null dimension in the state-space for which trajectories do not converge). A cooperative system with an irreducible Jacobian is strongly monotone. The irreducibility of the Jacobian is inherited from the irreducibility of the matrix $P^+ = [p_{ij}^{+}]$ given the form of the Jacobian and can therefore be checked graphically[2]. Systems whose time reversed version are cooperative are called competitive. The term cooperative comes from the fact that, for systems with a Metzler Jacobian, the increase of a state variable does not result in the decrease of another variable rate of accumulation. We will later see examples where this is not necessarily the case.

For compartmental systems, stronger results exist : (See [18] and [2] for a more rigorous formulation).

Property 2 (Rosenbrock). *A compartmental system with a non-singular Jacobian matrix $J(x)$ which is compartmental $\forall x \in \mathbb{R}_+^N$ has a unique equilibrium point which is globally and asymptotically stable if there exists a compact convex set $D \subset \mathbb{R}_+^N$ which is forward invariant.*

It is easy to verify that this result applies to (7) as

$$\begin{aligned} \frac{\partial f_i}{\partial x_j} &= \mu_j p_{ji}^{+} r'(x_j) & \geqslant 0 \\ \frac{\partial f_i}{\partial x_i} &= -\mu_i r'(x_i) & \leqslant 0 \\ \textstyle\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_j} &= -d_j \mu_j r'(x_j) & \leqslant 0 \end{aligned} \qquad (8)$$

where

$$r'(x) = \frac{\partial r(x)}{\partial x} = \frac{1}{(1+x)^2} \geqslant 0$$

The nonsingularity of the Jacobian matrix follows from the nonsingularity of the matrix P^+ which, once again, can be verified by inspection of the associated network. The existence of a forward invariant convex set can be shown if all the trajectories are bounded which will be the case if the traffic equation has a solution.

4. G-NETWORKS

G-Networks extend the traditional queueing theory paradigm in the sense that in addition to normal customers, signals are allowed to flow in the network. When a customer finishes service in queue i, it leaves the network with probability d_i, heads to queue j with probability p_{ij}^{+} or heads toward queue j as a signal with probably p_{ij}^{-} ($\sum_{j=1}^{N}(p_{ij}^{-} + p_{ij}^{+}) + d_i = 1$, $p_{ii}^{-} = 0$). When a signal arrives to a queue i, it triggers the instantaneous movement of a customer to queue j with probability q_{ij} or it forces the customer to leave the netwok with probability D_i ($\sum_{j=1}^{N} q_{ij} + D_i = 1$). Exogenous signals arrive at queue i as a Poisson process with rate λ_i. A signal arriving to an empty queue has no effect. The scope of G-Networks has been extended to the case where signals cause batch removals of customers as well as reset the queue to some pre-defined stationary values. G-Networks have been applied extensively (see [7] for a special issue on G-Networks). In this paper we restrict ourself to G-Networks with triggered customer movements.

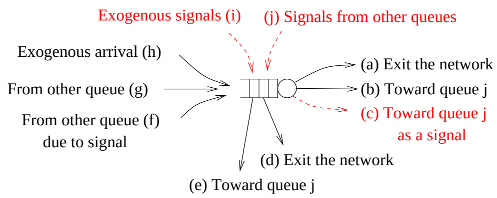

Figure 2: Fluid flow balance of customers and signals around a buffer i.

G-Networks, as Jackson's Networks, satisfy the product form property that can be stated as follows [6]

PROPERTY 3 (GELENBE). *Consider the non-linear system*

$$\begin{array}{rcl} \lambda_i^- & = & \sum_{j=1}^N \mu_j q_j p_{ji}^- + \lambda_i \\ \lambda_i^+ & = & \sum_{j=i}^N \mu_j q_j \left(p_{ji}^+ + \sum_{m=1}^N p_{jm}^- q_m q_{mi}\right) \\ & & + \sum_{j=1}^N \lambda_j q_j q_{ji} + \Lambda_i \end{array} \quad (9)$$

with $q_i = \frac{\lambda_i^+}{\mu_i + \lambda_i^-}$. *If there is a positive solution to (9) such that* $\lambda_i^+ < \mu_i + \lambda_i^-$ *then*

$$p(k) = \prod_{i=1}^N (1 - q_i) q_i^{k_i}$$

We now show once again that it is possible to design a compartmental system which, under the same conditions than those stated in Property 3, has a unique equilibrium point which corresponds to the average buffer occupancy which follows from Property 3.

Indeed, consider the (fluid) flow balance of customers and signals around a buffer i depicted in Fig. 2. The various quantities of interest are : (a) Flow of customers leaving the network after treatment $\mu_i r(x_i) d_i$, (b) Flow of customers heading toward queue j, $\mu_i r(x_i) p_{ij}^+$, (c) Flow of customers leaving the buffer as signals $\mu_i r(x_i) p_{ij}^-$, (d) Flow of ejected customers leaving the network $\lambda_i^-(x) r(x_i) D_i$, (e) Flow of ejected customers heading toward queue j, $\lambda_i^-(x) r(x_i) q_{ij}$. We also define the flow (f) $n_i^+(x)$ of customer arriving after having been ejected from another queue and the flow (g) $l_i^+(x)$ of customers arriving from another queue. The flow (h) Λ_i and (i) λ_i are the exogenous arrivals of customers and signals respectively and we finally define the flow (j) $l_i^-(x)$ of signal arrivals from other queues.

With these definitions, the flow balance of customer arrivals and departures may be written

$$\dot{x}_i(t) = -(\mu_i + \lambda_i^-(x)) r(x_i) + \Lambda_i + l_i^+(x) + n_i^+(x) \quad (10)$$

with

$$\begin{array}{rcl} l_i^-(x) & = & \sum_{j=1}^N \mu_j r(x_j) p_{ji}^- \\ l_i^+(x) & = & \sum_{j=1}^N \mu_j r(x_j) p_{ji}^+ \\ n_i^+(x) & = & \sum_{j=1}^N \lambda_j^-(x) r(x_j) q_{ji} \end{array}$$

and with $\lambda_i^-(x) = \lambda_i + l_i^-(x)$ and $\lambda_i^+(x) = \Lambda_i + l_i^-(x) + n_i^+(x)$. Consider the equilibrium point x^* of (10). By replacing in (10) $\lambda_i^-(x^*)$ by its value and posing $q_i = r(x_i^*)$ one obtains

$$0 = -(\mu_i + \lambda_i^-(x^*)) q_i + \lambda_i^+(x^*) \quad \Leftrightarrow \quad q_i = \frac{\lambda_i^+(x^*)}{\mu_i + \lambda_i^-(x^*)}$$

Furthermore, it is easy to see that

$$\begin{array}{rcl} \lambda_i^-(x^*) & = & \lambda_i + \sum_{j=1}^N \mu_j r(x_j^*) p_{ji}^+ = \lambda_i^- \\ \lambda_i^+(x^*) & = & \Lambda_i + \sum_{i=1}^N \mu_j r(x_j^*) p_{ji}^+ + \sum_{j=1}^N r(x_j) r(x_i) (\lambda_j + l_j^-(x^*)) \\ & = & \lambda_i^+ \end{array}$$

which shows our result. Moreover, system (10) may be rewritten in compartmental matrix form with compartmental matrix $G^G(x) = [g_{ij}(x)]$ defined as

$$g_{ii}(x) = -(\mu_i + \lambda_i^-(x)) \tilde{r}(x_i)$$

and

$$g_{ij}(x) = (\mu_j p_{ji}^+ + \lambda_j^- q_{ji}) \tilde{r}(x_j)$$

4.1 Competitive systems and inhibition in G-Networks

The Jacobian matrix of the system

$$\dot{x} = f^G(x) = G^G(x) x + \Lambda \quad (11)$$

has entries

$$\frac{\partial f_i^G}{\partial x_i} = -(\mu_i + \lambda_i^-(x)) r'(x_i)$$

and

$$\begin{array}{rcl} \frac{\partial f_i^G}{\partial x_j} & = & \mu_j (p_{ji}^+ - r(x_i) p_{ji}^-) r'(x_j) \\ & & +\left(\lambda_j + \sum_{m=1}^N \mu_m r(x_m) p_{mi}^+\right) q_{ji} r'(x_j) \\ & & + \sum_{m=1}^N \mu_j p_{jm}^- r(x_m) q_{mi} r'(x_j) \end{array}$$

It can be seen that non-diagonal entries of the Jacobian are no longer necessarily positive. However, we can state the following result:

PROPERTY 4. *If* $p_{ij}^+ > p_{ij}^- \ \forall i, j \in [1,, N], i \neq j$, *then, (11) is a cooperative system.*

This means that when this property and the irreducibility of the Jacobian are verified, and when system (11) has a unique equilibrium under the condition stated above, this equilibrium point is globally and asymptotically stable. However, when property (4) does not hold, very little can be said in general about the stability of (11). The presence of negative entries in the Jacobian are mathematical expressions of the inhibitory effects of signals in G-networks. In fact, G-Networks have been originally and successfully developed to model neural networks where inhibition plays an important role [5]. Therefore, this raises the important question

$$\begin{pmatrix} -(\mu_1+\mu_N r(x_N))r'(x_1) & 0 & \cdots & 0 & -\mu_N r(x_1) r'(x_N) \\ \mu_1 r'(x_1) & -\mu_2 r'(x_2) & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \cdots \\ & & \ddots & -\mu_{N-1} r'(x_{N-1}) & 0 \\ & & & \mu_{N-1} r'(x_{N-1}) & -\mu_N r'(x_N) \end{pmatrix} \quad (13)$$

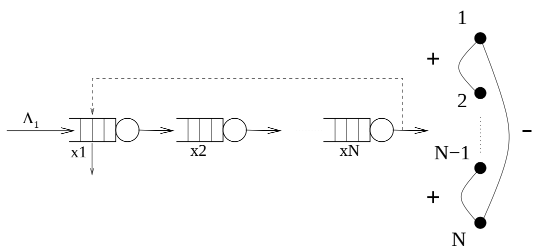

Figure 3: (Left) Example of a G-network with simple inhibition whose fluid-flow model is competitive. (Right) Sign structure of the Jacobian of the corresponding compartmental system.

of fluid-flow model stability. Chapman-Kolmogorov systems governing the dynamics of the state evolution of Continuous-time Markov Chains are known to be stable and such systems always reach equilibrium. However, this is a statistical equilibrium which is not incompatible with the idea of a fluid flow analysis that would predict instability. Instabilities in the form of sustained oscillations would perhaps simply reveal an increase in variance of the corresponding process or presence of periodicity in correlation function of the buffer occupancy point processes. Nevertheless, it should be stressed that there are no mathematical reasons, that we are aware of, for the stability of the approximate models to be related to some probabilistic characteristics of the corresponding queueing model. If this case was to be established, it would simply restrict the validity of the model (11) as a good dynamical approximation to the case where it is stable when the queueing network is itself stable.

4.2 Illustration

Let us consider the illustrative example shown in Fig. 3(left) made of a linear chain of M/M/1 queues. The output of the N^{th} buffer is transformed into signals heading back to the first queue. Each signal forces a queueing customer to leave the network. The compartmental system associated with this setup is as follows :

$$\left\{ \begin{array}{lcll} \dot{x}_1 & = & \Lambda_1 - (\mu_1 + \mu_N r(x_N)) r(x_1) & \\ \dot{x}_i & = & \mu_{i-1} r(x_{i-1}) - \mu_i r(x_i) & i = 2,,N \end{array} \right. \quad (12)$$

with Jacobian matrix (13) shown at the top of the page. The sign structure of (13) is shown in Fig. 3(right). Each node represents a state variable and an arc with positive (resp. negative) sign is drawn between node i and j if $\partial f_i/\partial x_j \geqslant 0$ (resp. $\leqslant$) or $\partial f_j/\partial x_i \geqslant 0$ (resp. $\leqslant$) for at least one $x \in \mathbb{R}_+^N$. It can be shown [19] that a system with sign symmetric Jacobian may be transformed into a cooperative system if no loop of its sign structure graph contains an odd number of negative signs. A system can be transformed into a competitive system if each loop of its sign structure graph contains an even number of positive signs. Therefore, system (12) is never cooperative and is competitive for odd values of N.

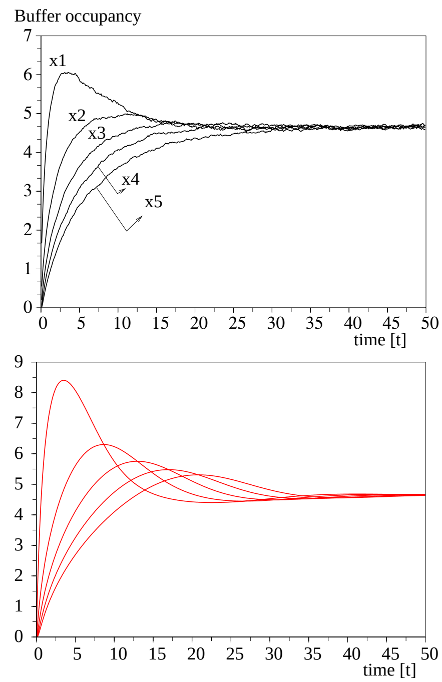

Figure 4: Comparison between the average of 10000 sample paths (Left) resulting from discrete event simulation and the solution (Right) of system (12).

The solution of (12) with $\Lambda_1 = 15$, $\mu_i = 10$, $N = 5$ is shown in Fig. 4(Right) and compared with results obtained with the discrete event simulator *omnet++*. The curves shown result from the ensemble average of 10000 sample paths.

If it can be argued that they are not exactly similar, the large overshoot is well reproduced and the time to settle to equilibrium is also correctly approximated which is already a fair contribution in regard to the simplicity of the system (12). Furthermore, it has been shown in [10] that the function $r(x)$ could be parametrised with the parameter a when rewritten so that $r(x) = x/(a + x)$ with $a > 0$ to take into account a larger range of traffic characteristics. In particular, when $a \to 0$, the system reacts deterministically and the queue length at equilibrium tends toward zero. This therefore motivates the introduction of the following correction

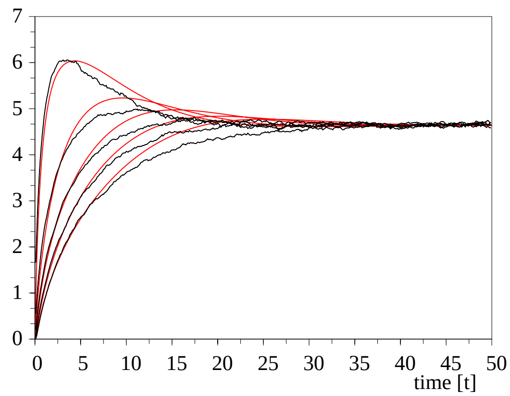

Figure 5: Solution of (12) with a correction term (14).

term

$$r(x) = \frac{x}{a+x} \quad a = 1 - 2A \cdot atan(B\dot{x})/\pi \qquad (14)$$

with $0 \leqslant A \leqslant 1$ which allows the output rate of a buffer to speed up and to decrease during non-steady conditions. Parameters A and B may be identified or calculated *a posteriori* to minimise a given criteria. Fig. 5 shows the solution of (12) corrected with (14) and with $A = 0.59, B = 0.7$. It can be seen that the approximation has improved drastically.

5. CONCLUSION

In this paper, a dynamical model that reproduces at equilibrium the behaviour of Jackon's and Gelenbe's networks has been presented. The existence of negative customers or inhibition in G-Network results in negative off-diagonal terms in the Jacobian matrix of the approximate system. Accordingly, the trajectories may exhibit a richer set of dynamical behaviours such as large overshoots as demonstrated in an illustrative example. It has been argued that the proposed approximate model is an adequate tool to study these transient phenomena. Furthermore, the proposed model being already used extensively in other disciplines of engineering sciences, the dialog between stochastic models and dynamical systems offers new ways of interpreting macroscopic (fluid-flow) behaviours in relation to microscopic (discrete event) interactions.

6. ACKNOWLEDGMENTS

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